CONSEQUENCES OF THEOREM OF THE CUBE:

Theorem (of the cube): Let X, Y be complete varieties, Z variety, $(x_0, y_0) \in X \times Y$ and $z_0 \in Z$. Let $\mathcal{L}$ be a line bundle on $X \times Y \times Z$ st $\mathcal{L}|_{\{x_0\} \times Y \times Z}, \mathcal{L}|_{X \times \{y_0\} \times Z}, \mathcal{L}|_{X \times Y \times \{z_0\}}$ then $\mathcal{L}$ is trivial.

Today:
(1) Any Abelian variety is projective.
(2) Any Abelian variety is a divisible group + study the structure of torsion points.

Corollary: If X is any variety, Y abelian variety $f, g, h : X \to Y$ and $\mathcal{L} \in \mathrm{Pic}(Y)$. Then $(f+g+h)^*\mathcal{L} \cong (f+g)^*\mathcal{L} \otimes (f+h)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}$.

Proof: We show that $M := \mathrm{LHS} \otimes (\mathrm{RHS})^{-1}$ is trivial. Note that

$$M = (f,g,h)^* \underbrace{(m^*\mathcal{L} \otimes m_{1,2}^*\mathcal{L}^{-1} \otimes m_{1,3}^*\mathcal{L}^{-1} \otimes m_{2,3}^*\mathcal{L}^{-1} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L} \otimes p_3^*\mathcal{L})}_{\tilde{M}}$$

with $m : Y \times Y \times Y \to Y$, $(p,q,r) \to p+q+r$; $m_{12} : Y \times Y \times Y \to Y$, $(p,q,r) \to p+q$; $p_i : Y \times Y \times Y \to Y$ projection.

Claim: $\tilde{M}|_{Y \times Y \times \{0\}}, \tilde{M}|_{Y \times \{0\} \times Y}, \tilde{M}|_{\{0\} \times Y \times Y}$ are all trivial.
This Claim $\Rightarrow M$ trivial by thm of the cube.

$\tilde{M}|_{Y \times Y \times \{0\}} \cong (m')^*\mathcal{L} \otimes (m')^*\mathcal{L}^{-1} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \otimes p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}$ is trivial $\square$. $m' : Y \times Y \to Y$, $(p,q) \longmapsto p+q$.

Definition: Let X be an abelian variety, $\mathcal{L} \in \mathrm{Pic}X$ define $\Phi_{\mathcal{L}} : X \longrightarrow \mathrm{Pic}X$, $x \longmapsto [T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}]$ with $T_x : X \to X$, $p \longmapsto p + x$.

We want to study $\ker \Phi_{\mathcal{L}} =: K(\mathcal{L})$.

Corollary: (thm of the square). Let $\mathcal{L} \in \mathrm{Pic}X$, $x, y \in X$ then $T_{x+y}^*(\mathcal{L} \otimes \mathcal{L}) \cong T_x^*\mathcal{L} \otimes T_y^*\mathcal{L}$.
i.e. $\Phi_{\mathcal{L}}(x+y) = \Phi_{\mathcal{L}}(x) \otimes \Phi_{\mathcal{L}}(y)$.

Proof: Apply the above corollary to $\mathcal{L}$ and $f : X \to \{x\}$, $g : X \to \{y\}$, $h = \mathrm{id}_X$. $\square$.

Proposition: $K(\mathcal{L})$ is a Zariski closed subgroup of X.
Proof: $T_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \cong (m^*\mathcal{L} \otimes p_2^*\mathcal{L}^{-1})|_{\{x\} \times X}$ by Seesaw thm:
$K(\mathcal{L}) = \{x \in X : T_x^*\mathcal{L} \otimes \mathcal{L}^{-1} = \Phi_{\mathcal{L}}(x) \text{ is trivial}\}$ is closed.

Application (1): Any abelian variety is projective.
We use the following criterion for ampleness,

Lemma: Let D be an effective divisor on X, $\mathcal{L}(D)$ the associated line bundle. TFAE,
(i) $H := \{x \in X \mid T_x^*D = D\} \subseteq K(L)$ is finite. ($T_x^*D = D$ as divisors).
(ii) $K(\mathcal{L})$ is finite
(iii) Linear system $|2D|$ is base point free and $X \xrightarrow{|2D|} \mathbb{P}^N$ is finite.
(iv) $\mathcal{L}$ is ample.

We now show that (i) is true for an appropriate D. Let $U \subseteq X$ be an affine, open subset, containing 0, $X \setminus U = D_1 \cup \cdots \cup D_n$ D_i irr. divisor then $D = \sum D_i$.

H is a closed subgroup of X hence H is complete and U is stable under translation by elements of $H \Rightarrow H \subseteq U \Rightarrow H$ is affine. $\Rightarrow H$ is affine + complete $\Rightarrow H$ is finite $\xRightarrow{\text{Lemma}}$ X is projective. $\square$

Proof of Lemma: (iii) $\Rightarrow$ (iv): use Serre's Cohomological criterion for ampleness: $H^i(X, \mathcal{O}(nD)\otimes\mathcal{F}) = 0$ $i>0$ $n>>0$ and compute $H^i(X, \mathcal{O}(nD)\otimes\mathcal{F})$ using a spectral sequence,

$$\begin{tikzcd} (iii) \arrow[r, Rightarrow] & (iv) \arrow[d, Rightarrow] \\ (i) \arrow[u, Rightarrow] & (ii) \arrow[l, Rightarrow, "\text{trivial}"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "|nD|=\varphi", "\text{finite}"'] \arrow[dr] & & \mathbb{P}^N \arrow[dl] \\ & pt & \end{tikzcd}$$

Grothendieck's spectral sequence

$H^p(\mathbb{P}^N, R^q\varphi_*(\mathcal{F}\otimes\mathcal{O}(nD))) \Longrightarrow H^{p+q}(X, \mathcal{F}\otimes\mathcal{O}(nD))$ however $R^q\varphi_*(\mathcal{F}\otimes\mathcal{O}(nD)) = 0$ for $q>0$ since φ is finite $\Rightarrow$ spectral sequence is degenerate on page 2.

(Slogan: pullback of ample via finite map is ample).

(iv) $\Rightarrow$ (ii) by contradiction $\mathcal{L}$ is ample, $K(\mathcal{L})$ not finite. $Y \subseteq K(\mathcal{L})$ positive dim. abelian subvariety $\mathcal{L}|_Y \otimes (-1_Y)^*\mathcal{L}|_Y$ is both ample & trivial, ample as $\mathcal{L}|_Y$ and $(-1_Y)^*\mathcal{L}|_Y$ are ample, trivial by the ~~thm. of the cube~~ (seesaw theorem) applied to $m^*\mathcal{L}|_Y \otimes p_1^*\mathcal{L}^{-1}|_Y \otimes p_2^*\mathcal{L}^{-1}|_Y$ + pull back along $Y \to Y\times Y$, $y \mapsto (y,-y)$

(i) $\Rightarrow$ (iii) Basepoint freeness: $\forall x\in X$ $T_x^*D + T_{-x}^*D \in |2D|$ by square thm.
Assume that H is finite and suppose by contradiction, $\varphi : X \xrightarrow{|2D|} \mathbb{P}^N$ is not finite.
Then some irr. curve $C \subseteq X$ is contracted to a point.
<u>idea</u>: from C, $\forall x_1, x_2 \in C$ $T^*_{x_1-x_2}D = D$ contradicts H being finite.

Application 2: X is a divisible group and for $n\geq 1$ $X_n = \ker(n\cdot x)$ is finite.

Corollary: For $n\in\mathbb{Z}$ $\mathcal{L}\in \mathrm{Pic}X$, $n^*\mathcal{L} \cong \mathcal{L}^{\otimes\frac{1}{2}(n^2+n)} \otimes (-1_X)^*\mathcal{L}^{\otimes\frac{1}{2}(n^2-n)}$

Proof: $(n+2)_X^*\mathcal{L} \otimes (n+1)_X^*\mathcal{L}^{\otimes(-2)} \otimes n_X^*\mathcal{L} \cong \mathcal{L}\otimes(-1_X)^*\mathcal{L}$ i.e. for $v_n = n_X^*\mathcal{L}$, $v_{n+2} - 2v_{n+1} + v$, this holds by corollary 2 applied to $(n+1)_X$, $g = 1_X$, $h = -1_X$ then by induction we are done.

Want to show, $\ker(n_X)$ is finite $\Rightarrow n_X$ is surjective, $\dim\ker n_X + \dim \mathrm{Im}\, n_X = \dim X$
$\mathcal{L}$ ample line bundle on $X \Rightarrow n_X^*\mathcal{L}$ is ample by above corollary. Also $n_X^*\mathcal{L}|_{\ker(n_X)}$ is trivial. $\square$

Proposition:
(1) $\deg n_X = n^{2g}$ $g = \dim X$ (2) n_X is separable $\Leftrightarrow p\nmid n$ $p = \mathrm{char}\, k$.
(3) $p\nmid n \Rightarrow X[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ $X[n] = \ker n_X$.
(4) $\exists i \in \{0,\dots,g\}$ $\forall n\geq 1$ $X[p^n] \cong (\mathbb{Z}/p^n\mathbb{Z})^i$

Reminder: finite surj. morphisms between complete varieties,

$$f : \begin{array}{c} X \\ \downarrow \\ Y \end{array} \Longrightarrow \begin{array}{c} k(X) \\ \uparrow \\ k(Y) \end{array} \qquad \begin{array}{c} k(X) \\ \uparrow \\ k(X)^{sep} \\ \uparrow \\ k(Y) \end{array} \qquad \begin{array}{l} \deg(f) := [k(X):k(Y)] \\ \mathrm{sepdeg}(f) = [k(X)^{sep} : k(Y)] \\ \mathrm{insepdeg}(f) = [k(X) : k(X)^{sep}]. \end{array}$$

Facts:
- If $D_1, \dots, D_n$ are divisors on Y ($n = \dim Y$) then $(f^*D_1 \cdots f^*D_n)_X = \deg(f)\cdot(D_1\cdots D_n)_Y$.
- $\# f^{-1}(y) = \mathrm{sepdeg} f$ for almost all y.
- Consider $\Omega^1_{k(Y)/k} \otimes_k k(X) \xrightarrow{\alpha} \Omega^1_{k(X)/k}$
  f is separable $\Leftrightarrow \alpha$ is injective.
  $k(Y) \subseteq k(X)^p \Leftrightarrow \alpha = 0$.

Proof: Compute $\deg(f)$: Choose D ample $E = D + (-1)^*D$ then $(-1)^*E = E$
$\Rightarrow n_X^*E \equiv n^2E$.
$\deg n_X (E \cdots E)_X = (n^2E, \dots, n^2E)_X = n^{2g}\overbrace{(E,\dots,E)_X}^{\neq 0 \text{ as } E \text{ ample}} \Rightarrow \deg n_X = 2g$.

$\# X[n] = \# \ker n_X = \#$ fiber of $n_X = \mathrm{sepdeg}(n_X)$. $p \nmid n \Rightarrow p \nmid \deg(n_X) \Rightarrow n_X$ is separable.
$\Rightarrow \# \ker n_X = n^{2g}$ for $p \nmid n$. We know that $\ker(n_X) = \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_r\mathbb{Z}$ $d_1 | \ldots | d_r$
Consider $\ell | d_1$ prime $\# X[\ell] = \ell^{2g} = \ell^r \Rightarrow r = 2g$.
Moreover $d_i | n$ so $\underbrace{d_1}_{\leq n} \cdot \ldots \cdot \underbrace{d_{2g}}_{\leq n} = n^{2g}$ hence $d_i = n$. $\forall i$.

$p | n \Rightarrow n_X$ is not separable. $\# X[p^n] = (p^m)^i$ $0 \leq i \leq g$.
$L = n_X^*(k(X)) \subseteq k(X)$
Show: $\Omega^1_{L/k} \otimes_L k(X) \xrightarrow{0} \Omega^1_{k(X)/k}$
(1) $d(n_X)_0 = n \cdot \mathrm{id}_{T_{X,0}} \equiv 0$.
(2) invariant differentials globally generate Ω^1_X

$L \subseteq k(X)^p \longrightarrow [k(X) : L] \geq p^g$: take $t_1, \ldots, t_g$ transcendence basis of $k(X)$.
t_i^j $1 \leq i \leq g$ $0 \leq j \leq p-1$ are lin. indep. over $k(X)^p \supseteq L$.